# Estimating the Lyapunov spectrum of time delay feedback systems from scalar time series

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On the basis of a recently developed method for modeling time delay systems, we propose a procedure to estimate the spectrum of Lyapunov exponents from a scalar time series. It turns out that the spectrum is approximated very well and allows for good estimates of the Lyapunov dimension even if the sampling rate of the time series is so low that the infinite dimensional tangent space is spanned quite sparsely.
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Nonlinear time series analysis became a common approach to the investigation of dynamical systems. On the one hand, this is due to the fact that the study of deterministic chaotic behavior became quite popular; on the other hand, the number and the reliability of available methods to explore this kind of systems did increase similarly rapidly (see, e.g., [1,2] and references therein). Collecting time series $\{x_n\}_{n=1}^N$ consisting of measurements of a single physical observable of the system, there are, at least for low-dimensional systems, methods that allow for good estimates of the fractal dimensions (usually the correlation dimension [3]) and the maximal Lyapunov exponent [4,5]. The reason why it is often sufficient to measure a scalar time series only is because of the theorems of Takens [6] and Sauer *et al.* [7]. Takens showed that, loosely speaking, one can reconstruct the unobserved variables by building so-called delay vectors $\vec{x}_i = (x_i, \ldots, x_{i-m+1})$. If the dimension $m$ of the delay vectors fulfils $m>2d$, where $d$ is the phase-space dimension of the system, then the space of the delay vectors is diffeomorphic to the original phase space. Sauer *et al.* generalized this theorem in the sense that they showed that if $m>2D_f$, where $D_f$ is the fractal dimension of the attractor, the new space is an immersion of the original attractor. Since the dimension and the Lyapunov exponents are invariant under diffeomorphisms, one can calculate these quantities in the delay space as well.

The restriction to low-dimensional systems is not due to theoretical, but to practical reasons. It was shown that the number of points needed for the analysis grows exponentially with the attractor dimension [8]. This makes it difficult or even impossible to handle high-dimensional objects. But a lot of interesting systems are rather high dimensional or even infinite dimensional. One example is spatially extended systems given by partial differential equations. Their phase space is indeed infinite dimensional and thus the attractors can be arbitrarily high in dimension. Another class of systems, and these are the ones we discuss in this paper, is the so-called time delay feedback systems [9]. In these systems one or more of the coordinates are fed back to the system with some time delay. In principle, one could imagine rather complicated mechanisms of the feedback, but we want to restrict ourselves to the case in which we have only one component fed back with one fixed delay time $\tau_0$. In this case the equations of motion read

$$\dot{\vec{y}}(t) = \vec{f}(\vec{y}(t), y^l(t-\tau_0)), \quad \vec{y} \in \mathbf{R}^d. \tag{1}$$

Due to the feedback of the $l$th coordinate the phase space of the system is the direct product of $\mathbf{R}^d$ with the space of the differentiable functions of the interval $[-\tau_0, 0[$.

Although the phase space is infinite dimensional, the system has a particular property if one uses the variables $[\dot{\vec{y}}(t), \vec{y}(t), y^l(t-\tau_0)]$. In this space Eqs. (1) define a set of $d$ constraints that restrict the system to lie on a $(d+1)$-dimensional manifold $\mathcal{M}$, which can be even smaller in dimension than the attractor. This property was used in [10,11] to determine the delay time $\tau_0$ and the vector field $f$ for the case $d=1$.

More general is the case $d>1$. Let $\{x_n\}_{n=1}^n$ be a time series of length $N$ and let $x_n$ be either the component of the system, which was fed back or an invertible function of it. Further, let $\tau_0$ be a integer multiple of the time intervals $\Delta t$ of the measurement $\tau_0 = p_0 \Delta t$. It is necessary to reconstruct the unmeasured variables of $\vec{y}(t)$. Recently we proposed a method [12] to reconstruct these variables on the above-defined manifold $\mathcal{M}$. Since the system is nonautonomous (in the sense that it is nonlocal in time), we borrowed ideas of Casdagli for input-output systems [13]. In this work Casdagli treated systems of the form

$$\vec{x}_{n+1} = \vec{f}(\vec{x}_n, \epsilon_n), \tag{2}$$

where $\epsilon_n$ is an arbitrary input. For such a system the Takens embedding cannot work if one only measures one variable. This is due to the fact that each measurement gives rise to a new $\epsilon_n$ and thus, a new source of "uncertainty." He argued that if, in addition to one variable, one also measures the inputs, a reconstruction in the sense of Takens's works again, but now with the tuples $(x_n, \epsilon_n)$. In our case the input is the delayed variable $x_{n-p_0\Delta t}$. So, in the spirit of Casdagli we build vectors (without loss of generality we set $\Delta t = 1$ and thus $\tau_0 = p_0$ for the rest of this paper),

$$\vec{v}_n(\tau_0) = (y_n, y_{n-1}, \ldots, y_{n-m+1}, y_{n-\tau_0}, \ldots, y_{n-\tau_0-m+1}), \tag{3}$$

where $m$ is the embedding dimension. In the sense of Takens this $m$ has generically to be at least $2d$ to give a diffeomorphic representation of the manifold $\mathcal{M}$.

 

In the embedding space the dynamics of the system, or in other words $\mathcal{M}$, is given through

$$x_{n+1}=g(\vec{v}_n(\tau_0)). \tag{4}$$

In principle the dynamics should also be given as a differential equation as in the original space. But due to the finiteness of $\Delta t$ and the resulting maplike structure of the time series, we treat it as a map. This, of course, could cause problems in the reconstruction. Since $\Delta t$ is finite, we are only able to span a space of dimension $\tau_0$ and we lose all the dimensions from smaller time scales. The only way to decide whether there was information lost, except when one has information about the typical time scales of the system, is to compare the results for different $\Delta t$ and to see if the results converge. We will come back to this point later on.

Usually we know neither the function $g$ nor the correct delay time $\tau_0$. So we have to make an ansatz to estimate both. Let $\tilde{g}_{\vec{a}}$ be an ansatz (e.g., polynomials, radial basis functions, or local linear models) of the function $g$ depending linearly on a set of parameters $\vec{a}$. Using a likelihood estimator for the optimal parameters $\vec{a}$ we can get $\tilde{g}$. And since the constraint (4) can only be fulfilled if we use the correct delay time, we can use the fit of the function also as a scheme to determine $\tau_0$ (for more details, see [12]).

Since we are interested in calculating Lyapunov exponents and thus in the dynamics in tangent space, we choose as an ansatz for $\tilde{g}_{\vec{a}}$ a local linear model

$$x_{n+1}=a_{0,n}+\vec{a}_n\vec{v}_n(\tau), \tag{5}$$

where the parameters $a_0$ and $\vec{a}$ depend on the position in space explicitly. Once we estimated the dynamics in the reconstructed space, we can introduce the dynamics in its tangent space. Therefore, we introduce the vectors $\vec{z}_n$ through

$$\vec{z}_n=(x_n,x_{n-1}, \ldots ,x_{n-\tau_0-m+1}). \tag{6}$$

The meaning of the vectors $\vec{z}_n$ is twofold. They both contain the delay reconstruction of the unobserved variable and they span the whole interval $[-\tau_0:0]$, which means the tangent space of the system. Of course, this is not the full tangent space that was infinite dimensional, but a discretized version of it. Thus, with a sampling time $\Delta t$ we can estimate $\tau_0/\Delta t$ Lyapunov exponents.

For $\vec{z}_n$ the Jacobians are simply given by

$$\mathbf{J}_{ij}^n=\left(\frac{\partial\vec{z}_{n+1}}{\partial\vec{z}_n}\right)_{ij}=\begin{bmatrix}\dfrac{dx_{n+1}}{dz_j}, & i=1\\ \delta_{i,j+1}, & i>1\,.\end{bmatrix} \tag{7}$$

The first row of the Jacobian contains the fitted coefficients $a_j$ at the positions of $(\vec{v}_n)_j$ and zeros elsewhere. The other rows define a shift operation. To obtain the exponents we iterate a set of vectors $\vec{e}_i$ in tangent space according to the dynamics given by the $\mathbf{J}^n$.

To demonstrate the strength of the method we present its application to numerical models. The first one is the Mackey-Glass equation [14],

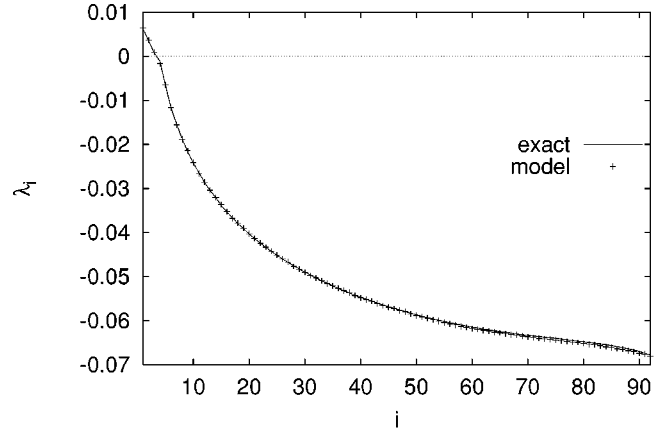


FIG. 1. Estimate of the first 92 Lyapunov exponents for Mackey-Glass with $\tau_0=46$. The symbols show the result for the iteration of $\mathbf{J}^n$; the solid line the result for the iteration of the exact tangent space dynamics with the same discretization of the tangent space.

$$\dot{y}=\frac{ay(t-\tau_0)}{1+y(t-\tau_0)^{10}}-by(t). \tag{8}$$

The parameters we chose are $a=0.2$, $b=0.1$, and $\tau_0=46$. Since this is a single component system, we do not need an embedding technique at all and we can work in the ''physical'' space. We used a time series of length 50 000 sampled with a $\Delta t$ of 1/2. This means we can estimate 92 exponents of the system. Figure 1 shows the estimate of these 92 Lyapunov exponents. The second curve, marked as exact, are the exponents obtained by iterating the exact tangent space dynamics and using the scheme of Farmer [15]. The discretization of the tangent space was the same with this scheme. One sees from the figure that the exponents agree nearly perfectly with the exact ones.

This is confirmed by Fig. 2 where we show the convergence of the Kaplan-Yorke [16] dimension as a function of the number of iterations. Note that the fit for the dynamics is done in a two-dimensional space, while the Kaplan-Yorke dimension is about 5. This means, using usual delay embedding we needed at least an embedding dimension of 6. In such a high-dimensional embedding space fitting a dynamics

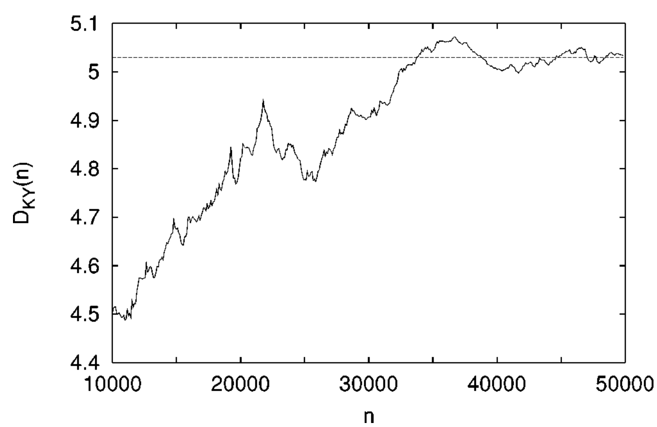


FIG. 2. Estimate of the Kaplan-Yorke dimension for the Mackey-Glass system as a function of the number of iterations. The horizontal line shows the exact value.

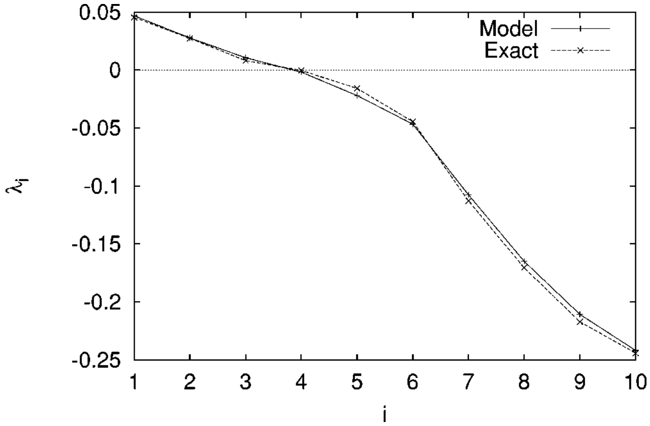


FIG. 3. Similar to Fig. 1 but for system (9).

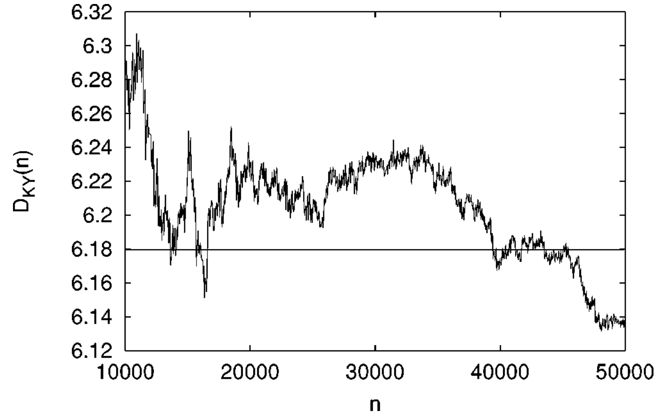


FIG. 4. Similar to Fig. 2 but for system (9).

is rather difficult and most probably the length of the time series had to be increased considerably. Even worse was the situation for the estimates of the Lyapunov exponents. Using, for example, the algorithm of Sano-Sawada [17] it was possible to estimate the exponents entering the Kaplan-Yorke formula, only. This is due to the fact that directions ''orthogonal'' to the attractor are not filled with data. Therefore, no information about the expansion rates in these directions is available, which means that it is impossible to estimate Lyapunov exponents corresponding to these directions. Or in other words, in linear space these directions would give Lyapunov exponents that are $-\infty$. But for curved manifolds the so-called spurious exponents appear, which can take any value. They can even be larger than the largest real one.

The second example we want to show is an extension of the Mackey-Glass equation to a two-dimensional system [18]. It is given by

$$\dot{y}_1(t) = \frac{a y_1(t-\tau_0)}{1+y_1^{10}(t-\tau_0)} + y_2(t),$$
$$\dot{y}_2(t) = -\omega^2 y_1(t) - \rho y_2(t), \tag{9}$$

with the parameters $a=3$, $\omega^2=2$, $\rho=1.5$, and $\tau_0=5$. The observed variable is $x_1$ sampled with $\Delta t=1/8$ and again measuring a time series of length 50 000. Since Eq. (9) defines a two–dimensional system we now have to use a delay embedding to reconstruct the manifold. In [12] we showed that $m=2$ is sufficient for this system.

Figure 3 shows the first ten Lyapunov exponents for this system. Although in principle we could determine 40 exponents from the data, we estimated only the first ten to save CPU time. Also here the agreement with the exact values is very good. Note that since we now have to work on a four-dimensional manifold, we cannot expect that the local linear model is as good as in the single component system (8). This is due to the fact that we need larger neighborhoods to find enough neighbors for the fit. This means that errors due to curvature effects are stronger.

Figure 4 shows, similarly to Fig. 2, the evolution of the Kaplan-Yorke dimension for system (9). Although the value is not perfect the deviation from the exact value is still smaller than 1%. The deviation is, as already mentioned, due to the fact that we have to work in a four–dimensional space. This leads to a stronger influence of curvature effects. One could try to overcome this problem by using a global model or long time series. But, such an approach is out of the scope of this paper.

Let us now address the question of the effect of the finite sampling. As already mentioned, one loses information, since only $\tau_0/\Delta t$ dimensions of the originally infinite dimensional space are accessible from the time series. We can only expect to get reasonable results if $\Delta t$ is smaller than the fastest time scale. If, on the other hand, $\Delta t$ is too large, we will change the apparent properties of the system and this should also be visible in the Lyapunov spectrum. Figure 5 shows the results of the calculation of the Lyapunov exponents obtained from system (8) as a function of the ''dimensional discretization'' (coarse graining) in tangent space. One sees that for taking into account more and more directions the results converge. This discretization corresponds to a finite sampling time. This means that we can use the convergence properties of the Lyapunov spectrum, as a function of the sampling time $\Delta t$, to check whether the results are reliable or not.

Figures 6 and 7 show the first ten Lyapunov exponents estimated from time series of length 50 000 of the systems (8) and (9), respectively. The different curves correspond to different values of the sampling time. One clearly sees that for sufficiently small values of $\Delta t$ the curves coincide, while for too large $\Delta t$s the results differ strongly from the

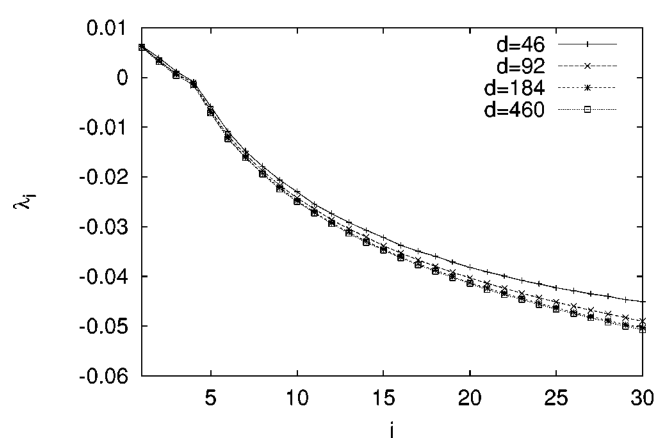


FIG. 5. First 30 Lyapunov exponents of system (8) for different values of the ''dimensional discretization'' of the tangent space.

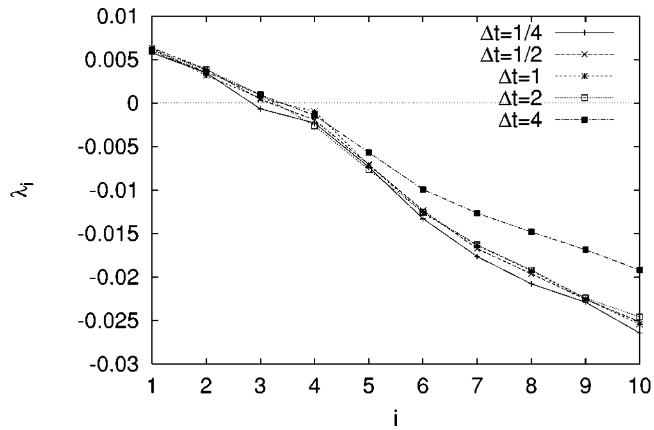


FIG. 6. First ten Lyapunov exponents of system (8) for different values of $\Delta t$.

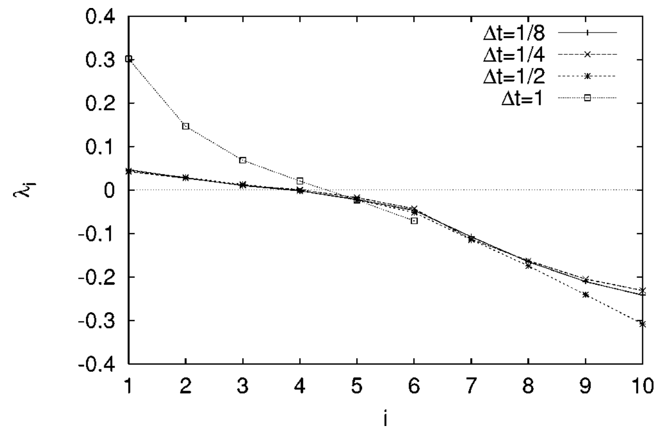


FIG. 7. Similar to Fig. 6 but for system (9).

asymptotic result. These results show that we can use the estimates of the spectrum to decide whether the given time series contains all the information we need to quantify the chaotic properties of the system.

On the basis of a recently developed method to model time-delayed feedback systems, we presented a technique to estimate the spectrum of Lyapunov exponents of such systems. It turns out that the estimated spectrum gives reasonably good values of the exponents as long as the sampling time is sufficiently small to resolve all relevant time scales of the system. Since the embedding technique is independent of the dimension of the attractor we are able to work in quite low-dimensional spaces, which allows for good estimates with reasonable lengths of the time series, even in the case where the attractor's dimension is very high. We demonstrated the efficiency of this method by applying it to two numerical systems. An extensive analysis of an experimental system using this method will be presented elsewhere [19].

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